Unification of self-thinning models for forestry

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Reminder: what is self-thinning?

Due to competion, as a forest grows, the number of trees it contains decreases. This phenomenon is called **self-thinning**.

In 1933, Reineke remarked that the plot of the number of trees per surface unit versus the average diameter of trees in a log-log scale seemed to follow a straight line.

$$\log N = k - a \log D$$

This remark is mainly for **monospecific** and **even-aged** stands.

He used that observation to build a Stand Density Index (SDI) allowing to compare stands at different development stages:

$$SDI = N\left(\frac{D}{10}\right)^{a} \text{ (}D\text{ in inches)}$$

Curtis (1970) used this idea to build various density indices, including a Relative Density Index:

$$RDI = {}^{N}/_{N_{max}}$$

Where N is the current stand density and N_{max} the theoretical maximal number of trees for a stand at the same development stage.

In fact, the previous formulations are not for the self-thinning trajectory but only for its limit: the **maximum density line**.

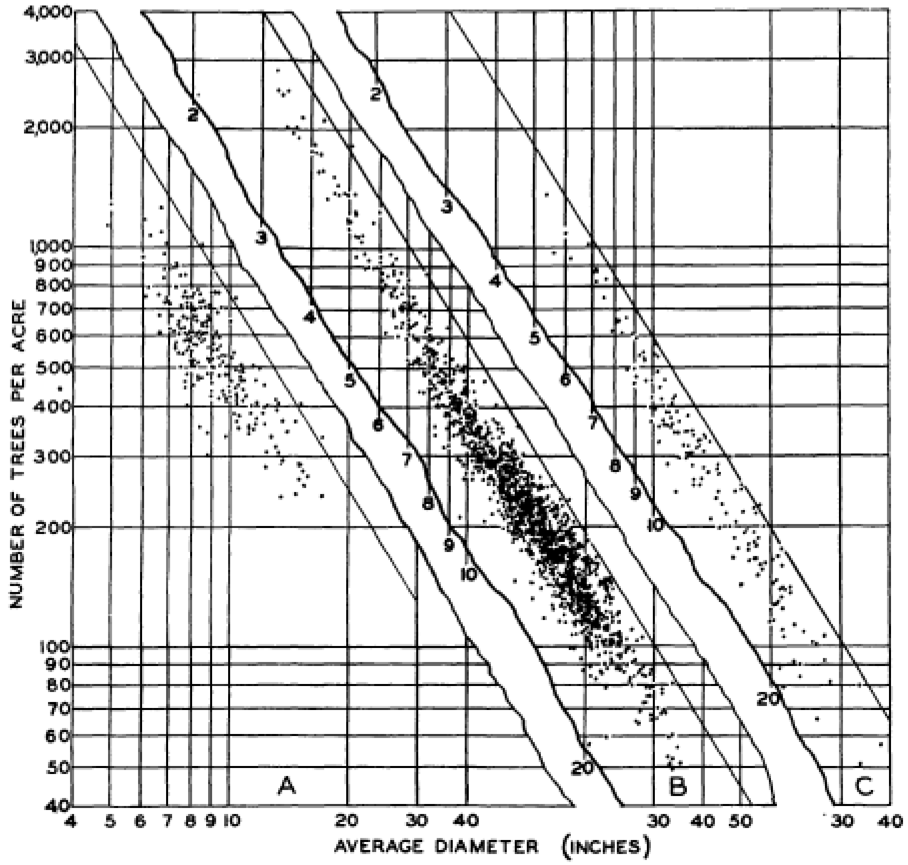


FIGURE 4.—Maxima curves for: A, Mixed conifer stands in California; B, Douglas fir in Washington and Oregon; C, Douglas fir in northern California. Note that the maximum stand-density index is almost identical (approximately 595) for both groups of Douglas fir

Reineke, 1933

Real self-thinning trajectories

The trajectory followed by real stands (in the log-log space) is not a straight line (Hozumi, 1977, 1980, 1983), but a curve that tends to a straight line (Kurinobu et Miyaura, 2011).

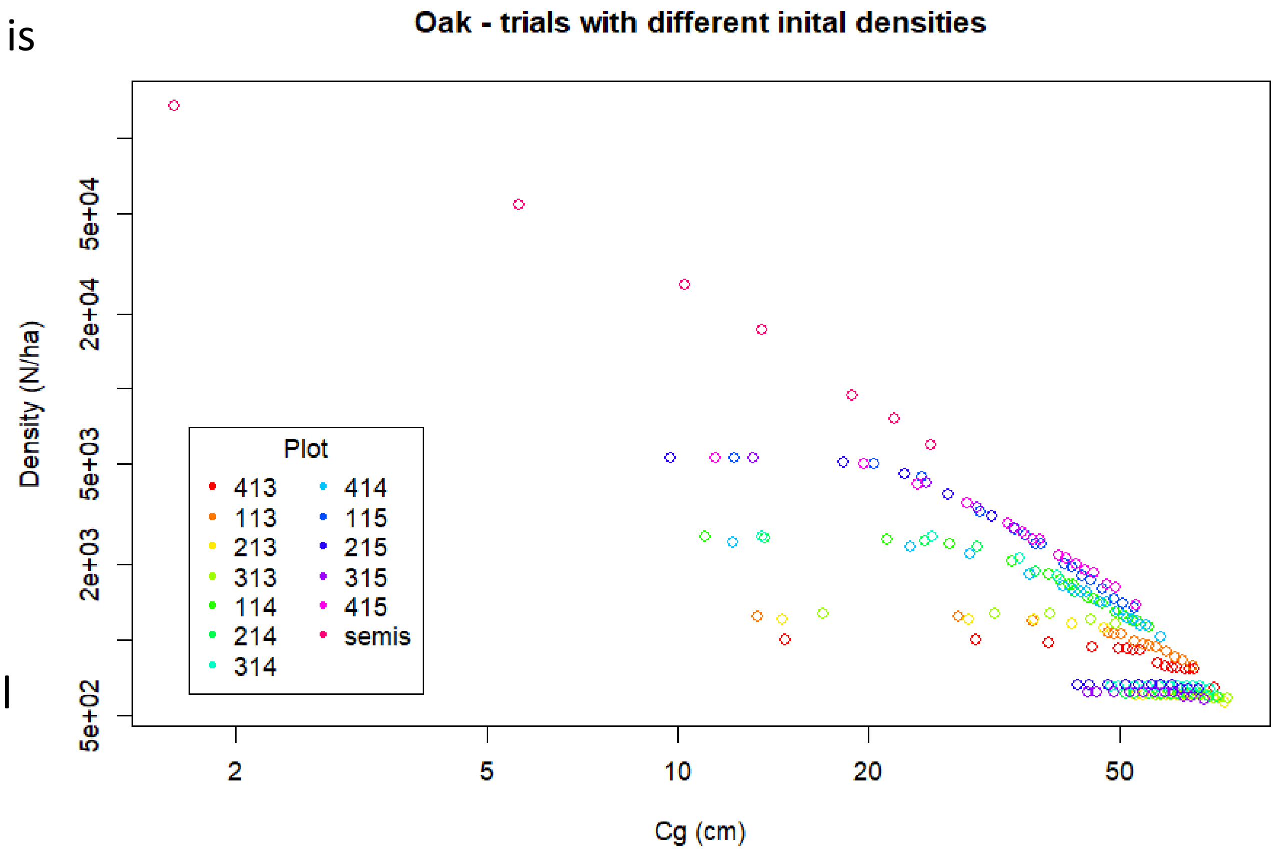
A first model has been proposed by Nilson (1973):

$$N = N_{25}\left(\frac{25+k}{D+k}\right)^2 \quad (D \text{ en cm})$$

Where N_{25} is the number of trees in the stand at the stage $D = 25cm$.

If $k = 0$, this is almost Reineke's SDI.

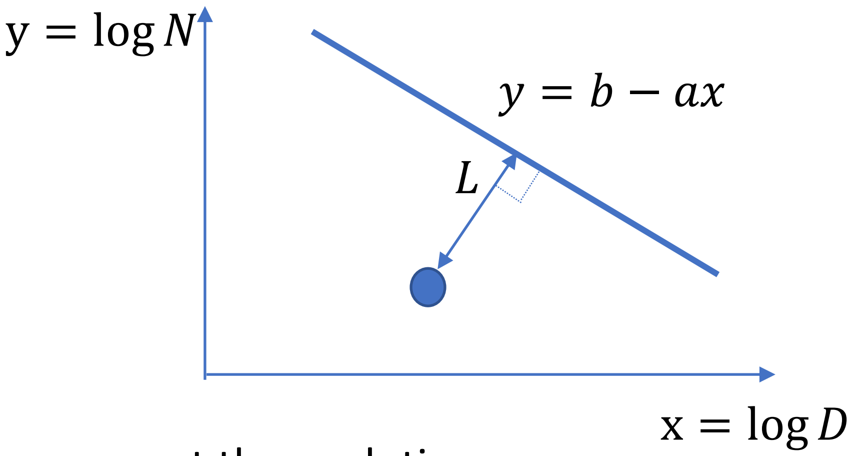
In 2016, Ningre, Ottorini and Le Goff built a parabolic model defined by two contact points, one on each of the two asymptotes. Between those points, the curve is a piece of parabola which slope at each contact points is equal to the slope of the corresponding asymptote.



Data from an experiment at Lyons-la-Forêt (Northern France)

Kurinobu and Miyaura's approach

In 2006, Kurinobu and Miyaura proposed an approach based on the Euclidian distance L between a point of the trajectory and the Maximum Density Line (in log-log scale):



$$r = \frac{\Delta y}{\Delta x} = b_1[1 - b_2 \exp(-b_3 L)] - 2.0$$

where b_1, b_2 and b_3 are to be estimated. The quantities Δx et Δy respectively represent the evolution of $\mathrm{x} = \log D$ and of $\mathrm{y} = \log N$ between two successive dates of observation.

In 2011, Kurinobu and Miyaura prefer a more simple expression :

$$r = \frac{\Delta y}{\Delta x} = a \cdot \exp(-c \cdot L)$$

where a and c are parameters to be estimated.

Remarks about Kurinobu and Miyaura's approach

Kurinobu and Miyaura use the distance (in the log-log plane) between a point belonging to the trajectory and the distance L of this point to the maximum density line.

Hence they use the Euclidian distance between two points in the plane.

The distance between the points $P_1 = (D_1, N_1)$ and $P_2 = (D_2, N_2)$ is:

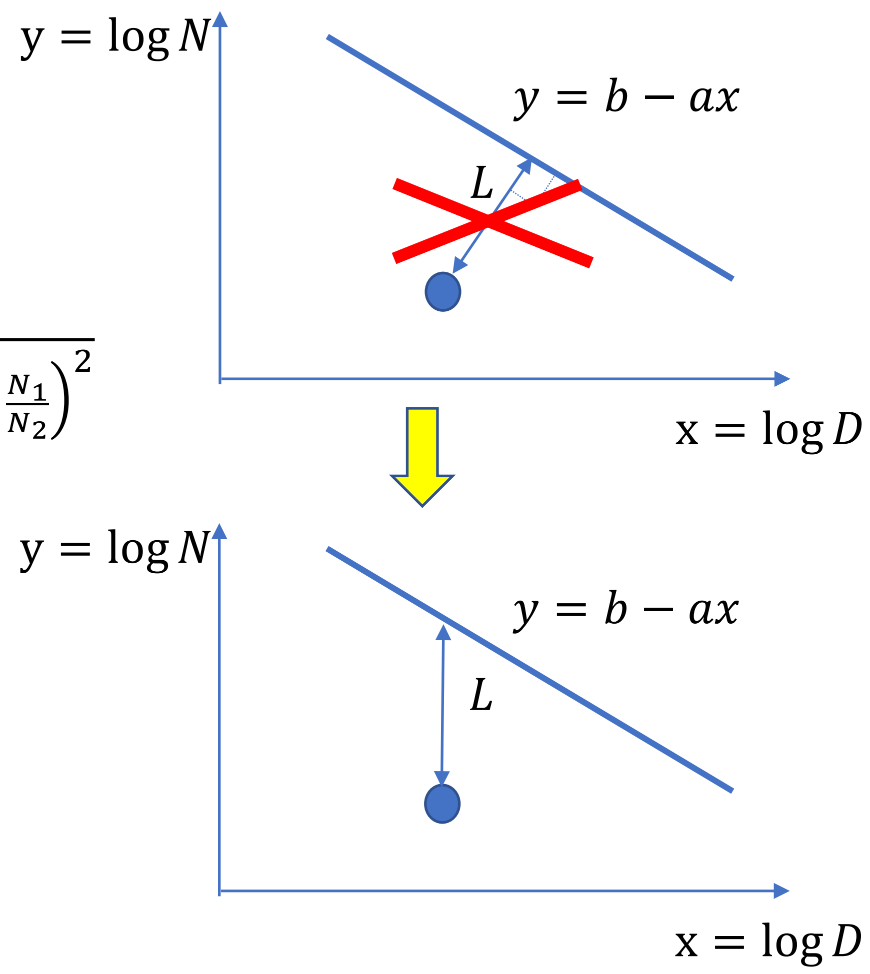
$$\mathrm{d}(P_1, P_2) = \sqrt{(\log D_1 - \log D_2)^2 + (\log N_1 - \log N_2)^2} = \sqrt{\left(\log \frac{D_1}{D_2}\right)^2 + \left(\log \frac{N_1}{N_2}\right)^2}$$

Hard to understand what that distance really means!

Idea: Define L as the length of the **vertical** segment from the trajectory point to the maximum density line.

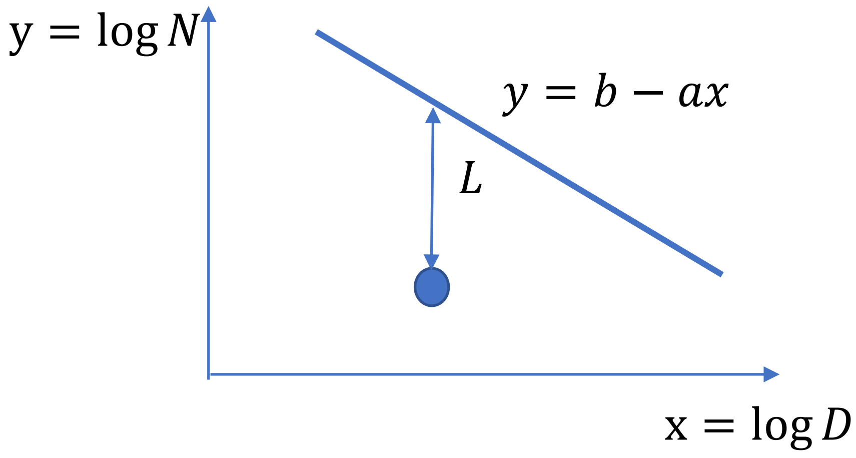
We obtain: $L = y_{max} - y = \log \frac{N_{max}}{N} = -\log RDI$

Another suggestion: replace $r = \frac{\Delta y}{\Delta x}$ by $r = \frac{dy}{dx}$ to work with continuous time



Generalised approach

We study the differential equation : $\frac{dy}{dx} = -a \cdot f(L)$



where f is a function that does not depend to the maximum density line parameters and that additionnally has the following properties:

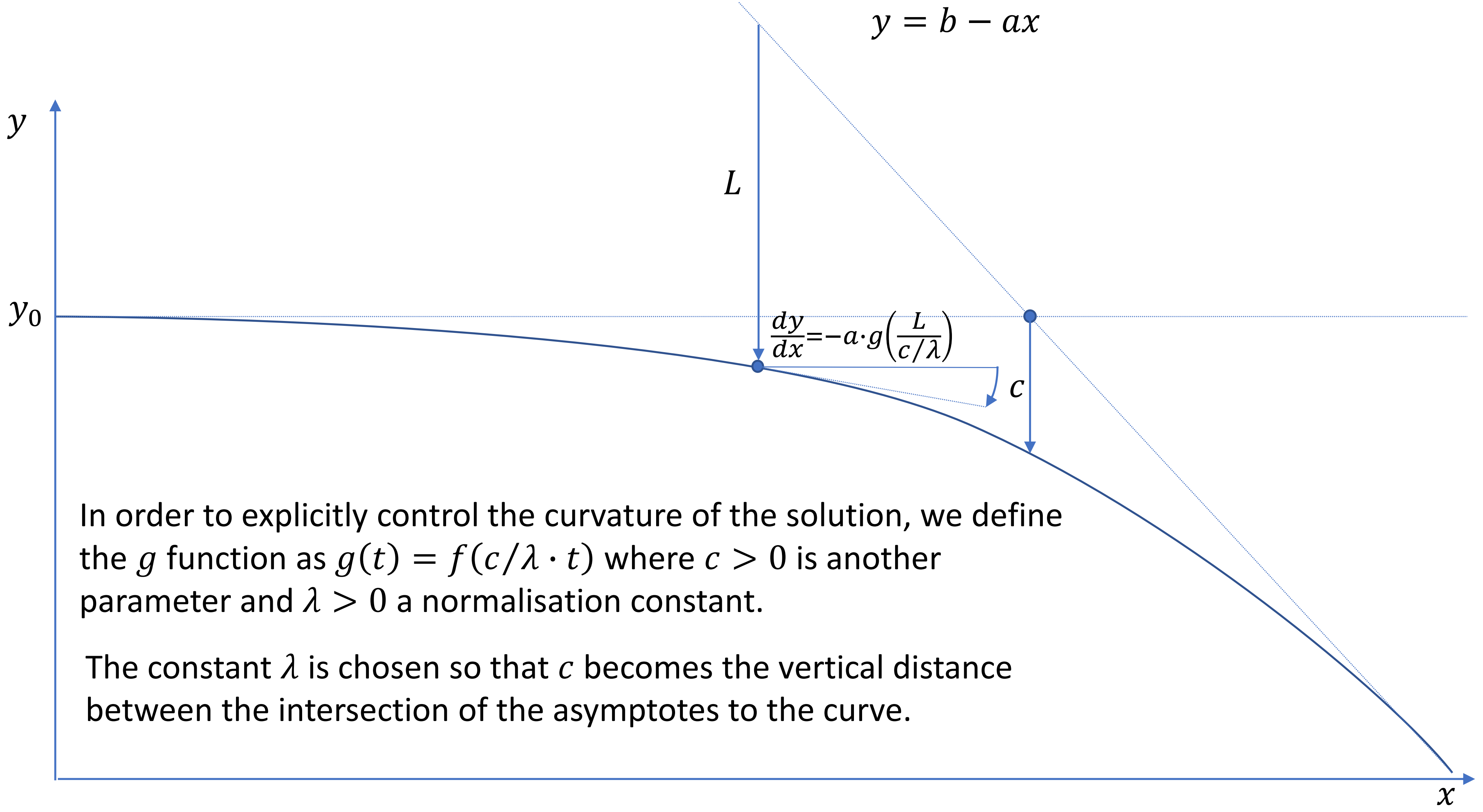
- f is defined on $\mathbb{R}^+$;
- $f(0) = 1$; f is continuous and decreasing, $\lim_{L \to +\infty} f(L) = 0$;
- $\int_0^{+\infty} f(L)dL < +\infty$.

It can then be shown that the curve that represents the solution of the differential equation:

- Is decreasing ;
- Does not admit an inflexion points;
- Admits an horizontal asymptote at $-\infty$;
- Admits the maximum density line as an asymptote at $+\infty$;
- Is located under its asymptotes.

→ The curve looks like an hyperbole.

Illustration of the modified approach



In order to explicitly control the curvature of the solution, we define the g function as $g(t) = f(c/\lambda \cdot t)$ where $c > 0$ is another parameter and $\lambda > 0$ a normalisation constant.

The constant λ is chosen so that c becomes the vertical distance between the intersection of the asymptotes to the curve.

Solution of the differential equation

If (x_0, y_0) is a known point of the trajectory:

With $L_0 = b - ax_0 - y_0$

$$y = b - ax - (c/\lambda) \cdot h^{-1}\left(h\left(\frac{L_0}{c/\lambda}\right) - \frac{a}{c/\lambda}(x - x_0)\right)$$

If $x_0 \to -\infty$, $y_0 = \lim_{x \to -\infty} y(x)$:

$$y = b - ax - (c/\lambda) \cdot h^{-1}\left(\alpha - \frac{b - ax - y_0}{c/\lambda}\right)$$

Where :

- $h(t) = \int \frac{dt}{1 - g(t)}$;
- $\alpha = \lim_{t \to +\infty}(h(t) - t)$;
- $\lambda = h^{-1}(\alpha)$.

The normalisation constant λ is defined so that the c parameter becomes the length of the vertical distance between the curve and the intersection of its asymptotes.

To each g function is associated a normalised function g_0 defined as $g_0(t) = g(\lambda t)$, with associated h_0 and α_0. We have then $\lambda_0 = h_0^{-1}(\alpha_0) = 1$.

Some particular cases

- $g(t) = \frac{1}{1+t^2}$ ⇨ $$y = \frac{b - ax + y_0 - \sqrt{(b - ax - y_0)^2 + 4(c/\lambda)^2}}{2}$$ $\lambda = 1$

  Hyperbolic model

- $g(t) = e^{-t}$ ⇨ $$y = b - ax - (c/\lambda) \cdot \log\left(1 + \exp\left(\frac{b - ax - y_0}{c/\lambda}\right)\right)$$ $\lambda = \log 2$

  Nilson-Kurinobu-Miyaura model ("NKM")

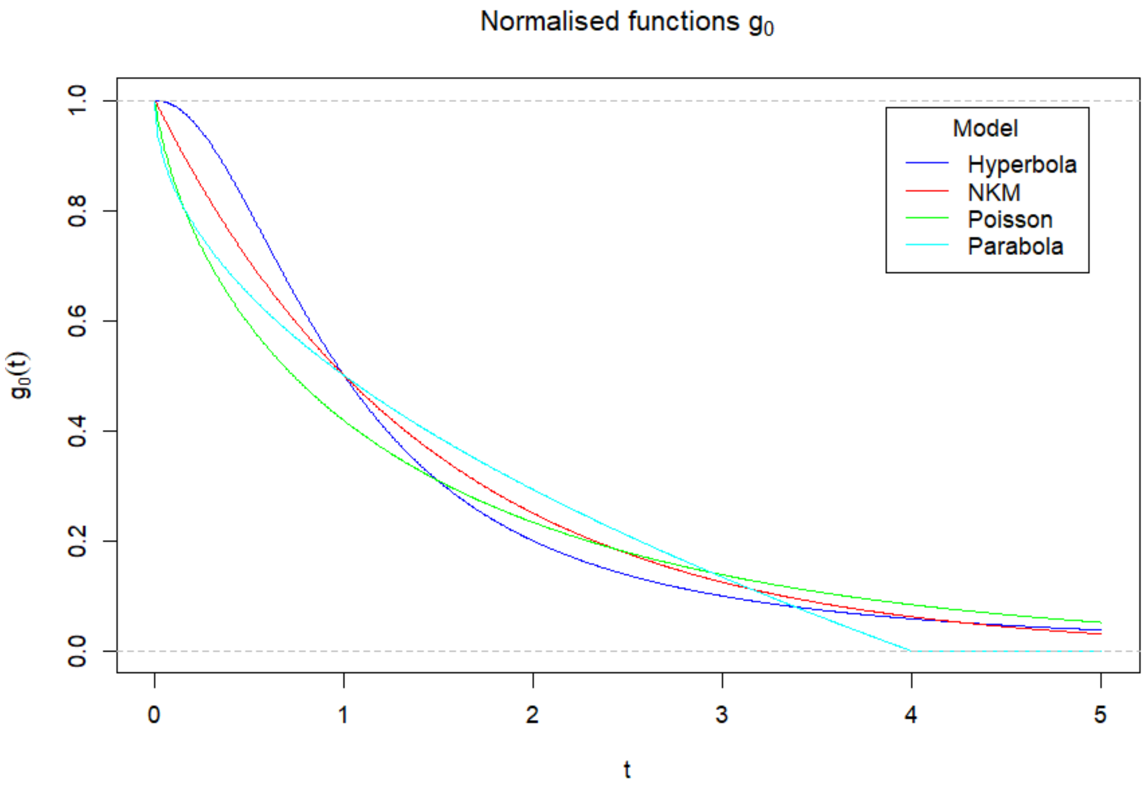
- $g(t) = 1 + (e^t - 1) \cdot \log(1 - e^{-t})$ ⇨ $$y = b - ax + (c/\lambda) \cdot \log\left(1 - \exp\left(-\exp\left(\frac{b - ax - y_0}{c/\lambda}\right)\right)\right)$$

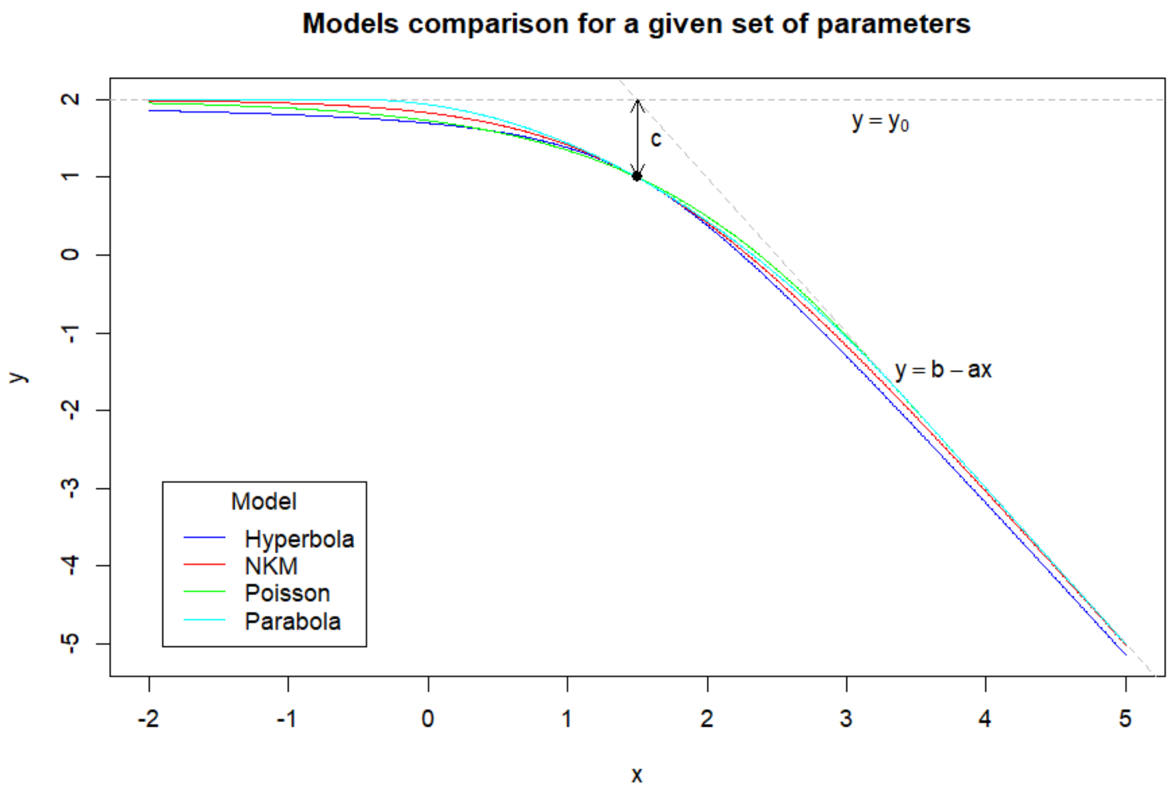
  $\lambda = -\log(1 - e^{-1})$ **Poisson model**

- $\begin{cases} g(t) = 1 - \frac{1}{2}\sqrt{t} & if\ 0 \le t \le 4 \\ g(t) = 0 & otherwise \end{cases}$ ⇨ $$\begin{cases} y = y_0 & if\ x \le x_0 \\ y = y_0 - \frac{1}{c/\lambda}\left(c/\lambda - \frac{b - y_0 - ax}{4}\right)^2 & if\ x_0 \le x \le x_1 \\ y = b - ax & if\ x \ge x_1 \end{cases}$$

  $x_0 = \frac{b - y_0 - 4c/\lambda}{a}$, $x_1 = \frac{b - y_0 + 4c/\lambda}{a}$, $\lambda = 1$ **Parabolic model (Ningre, Ottorini & Le Goff, 2016)**

Graphical representation





Despite differences between the g_0 functions, the resulting trajectories look very similar.

Application to simulated data

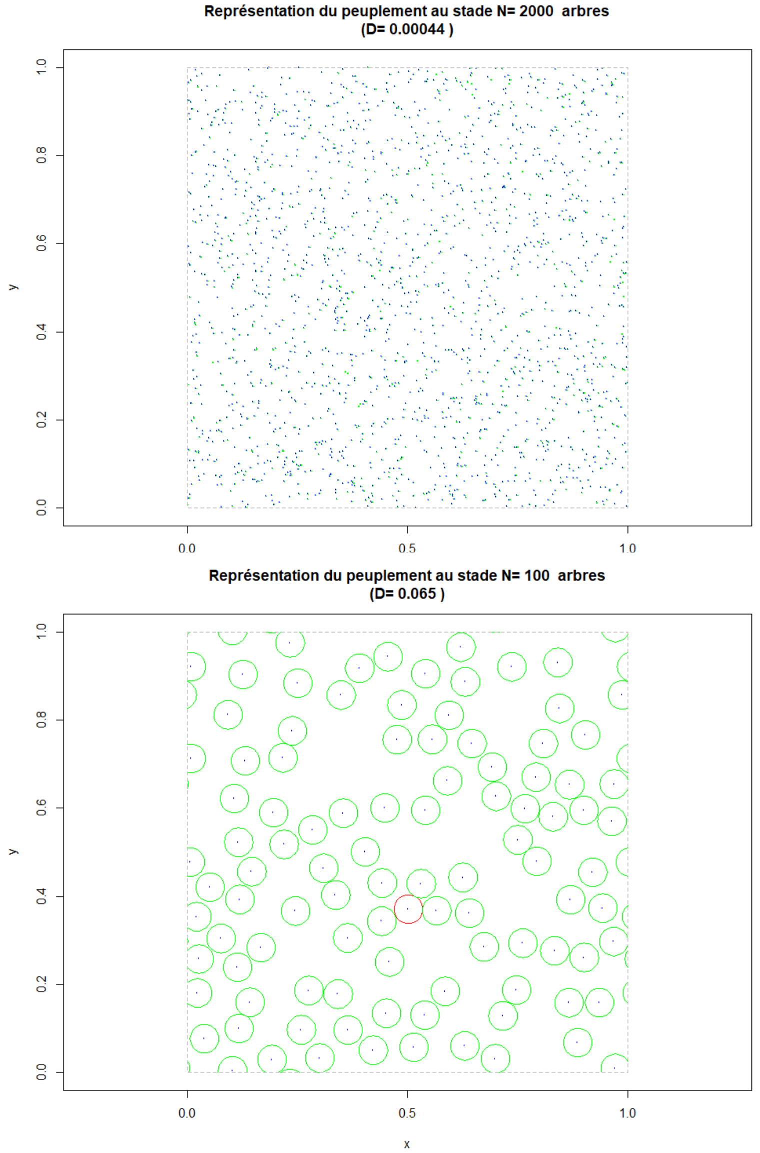
We define a 2-D toric space,

N_0 initial points are distributed on it ;

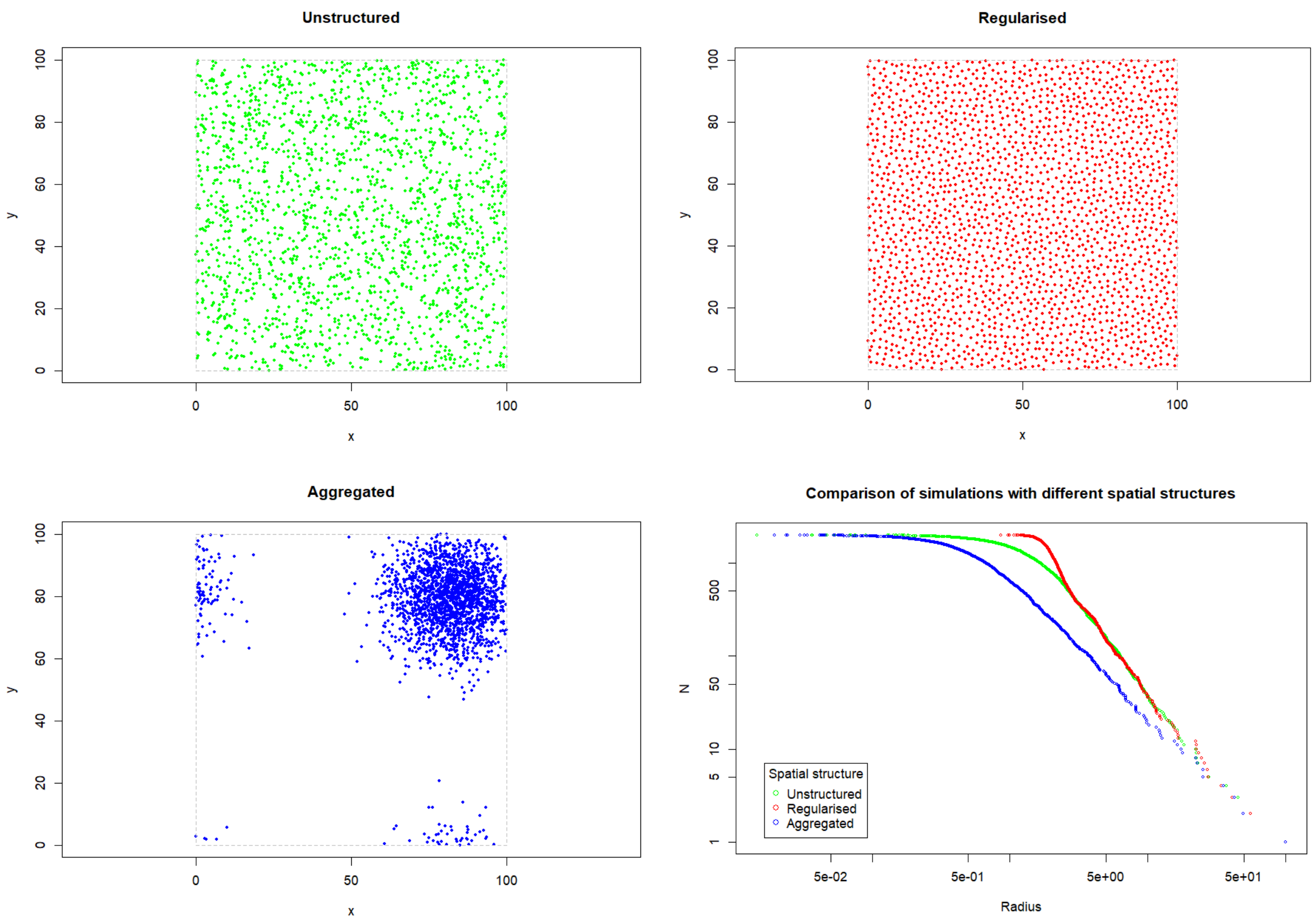
- All these point “grow” at the same speed;
- As soon as two circles touch each other, one is randomly eliminated (Bernoulli sampling);
- Until only one circle remains.

We study the couples N-D (Remaining circles – current diameter).

Initial points are distributed according different spatial structures (unstructured, aggregated, regularised).



Comparaison of several spatial structures



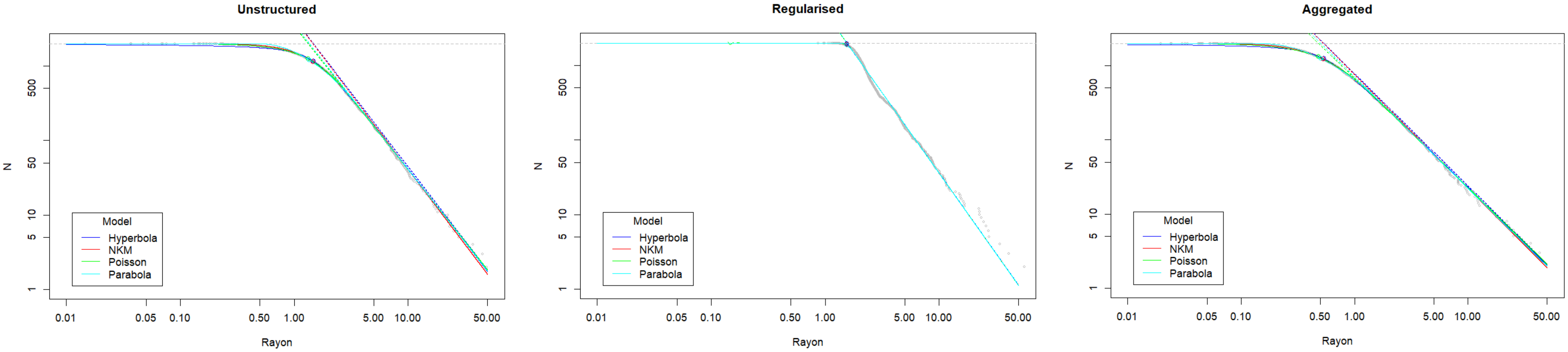
Here, $N_0 = 2000$

- Trajectories have expected shapes (hyperboloid)
- The asymptotes corresponding to the maximum density lines are very comparable
- Main differences are linked to the curvature

When the points begin to be « in competition ».

Fitting of models to the trajectories

N_0 is known so it is not estimated. Parameters a, b et c are estimated.



- In all cases, no problem to fit the models.
- For a given simulation, estimated parameters from a model to the others are very close to each other;

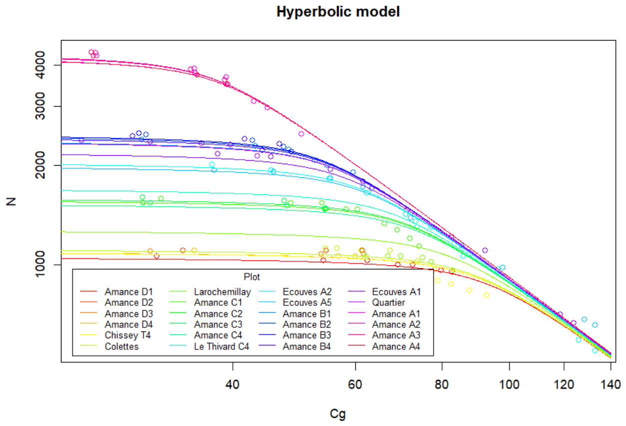
⇨ The evaluated models are hardly distinguishable from each others.

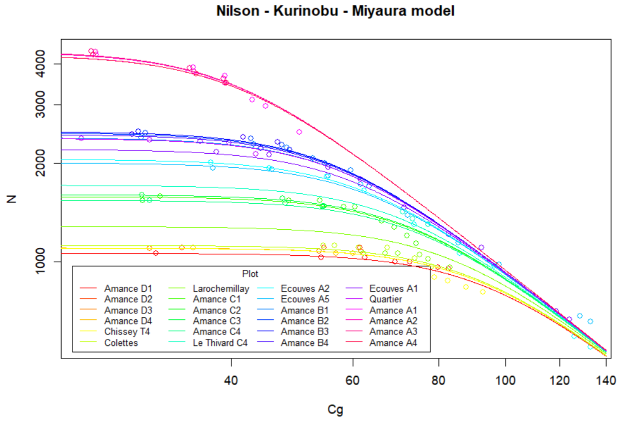
In fact, they differ by the speed at which then tend to the asymptotes $(0, \frac{k}{x}$ *or* $e^{-|kx|}$ *according to the model)*

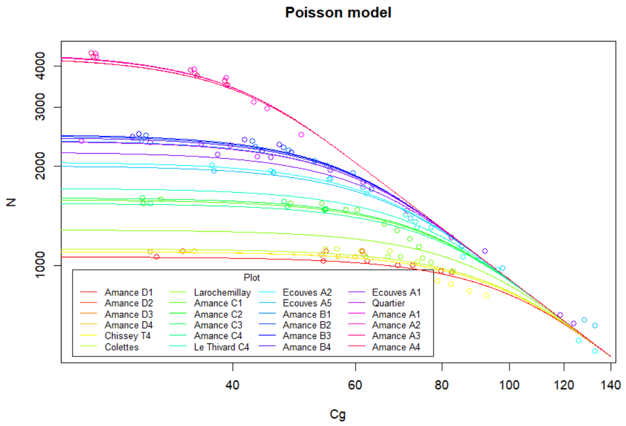
Application to forestry data

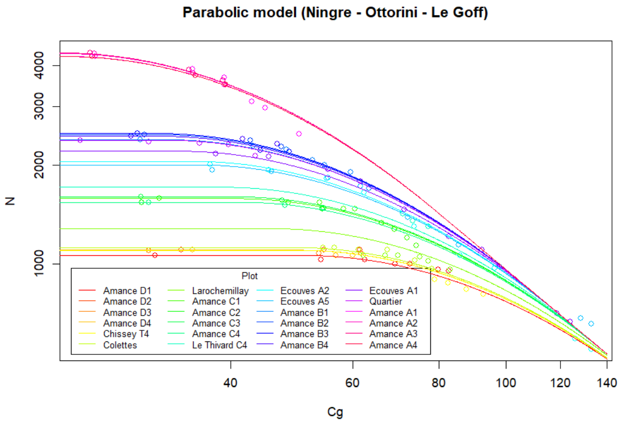
All the data used in this presentation have been compiled by François Ningre (Inrae Nancy)

Trials on Douglas-fir (*Pseudotsuga menziesii* (Mirb.)) with different initial plantation densities (GIS-Coop and Lerfob networks)

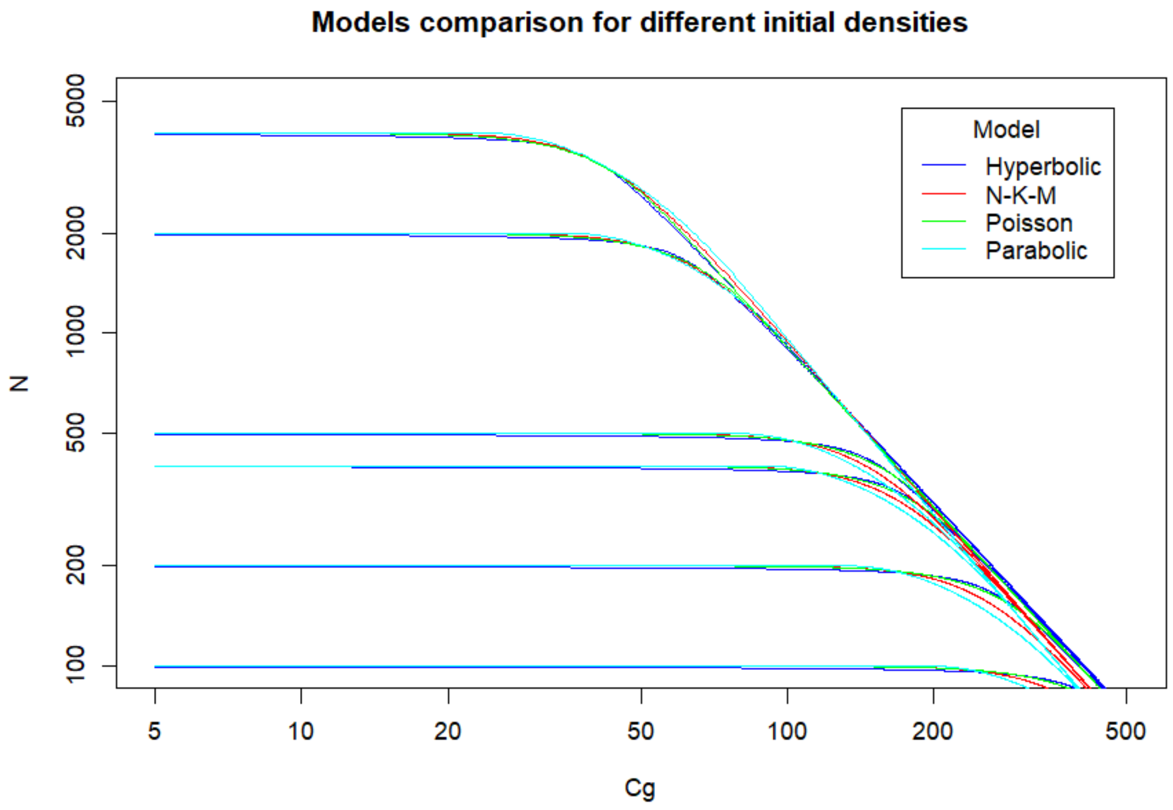








Comparison of Douglas-fir models for a same initial density



Same remarks than with simulated data:

- No difficulty to fit the curves;
- Models are hardly distinguishable from each others: parameters a, b and c are quite the same from one model to the other.

According to the situation (real or simulated data, species or spatial structure), the best model is not always the same.

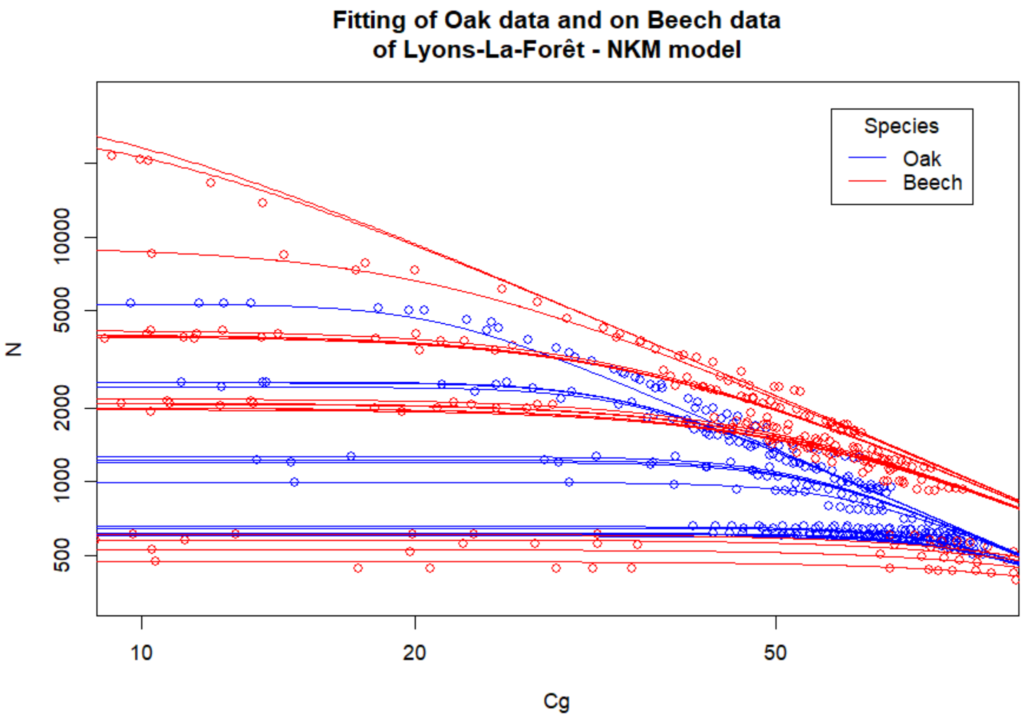
None of the tested models appears to be really better than the others.

For a given model and fixed a, b and c parameters, curves with different y_0 are just translated from each other according to a vector of slope $-a$.

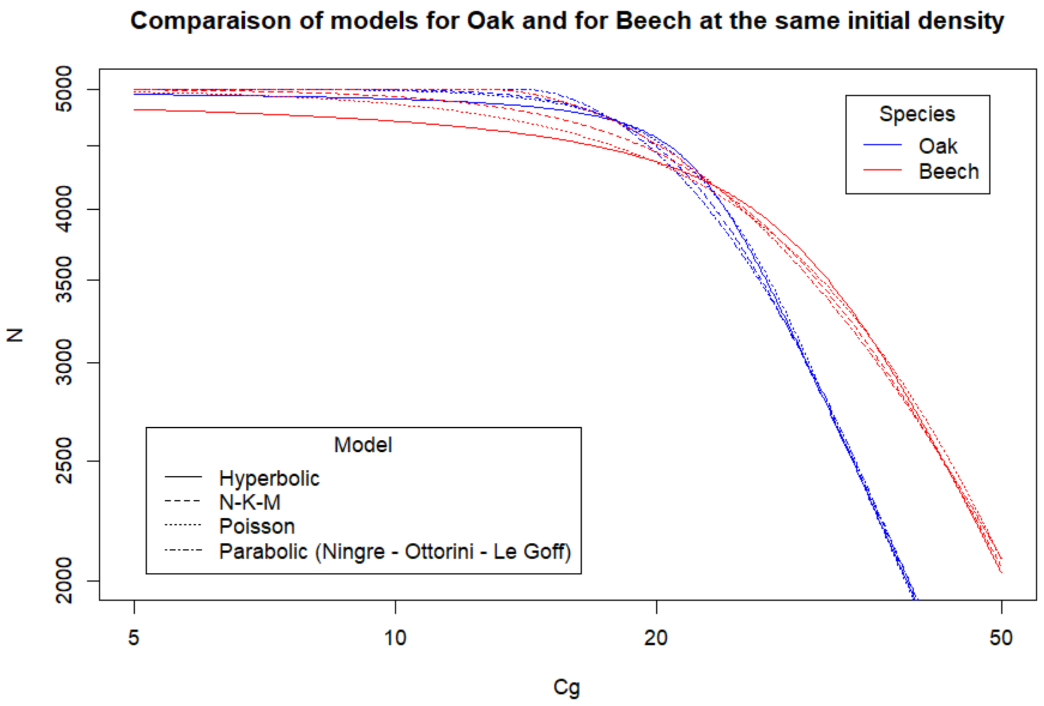
Oak-Beech comparison at Lyons-La-Forêt

Pedo-climatic conditions are comparable between the following monospecific Oak and Beech trials.

Real data



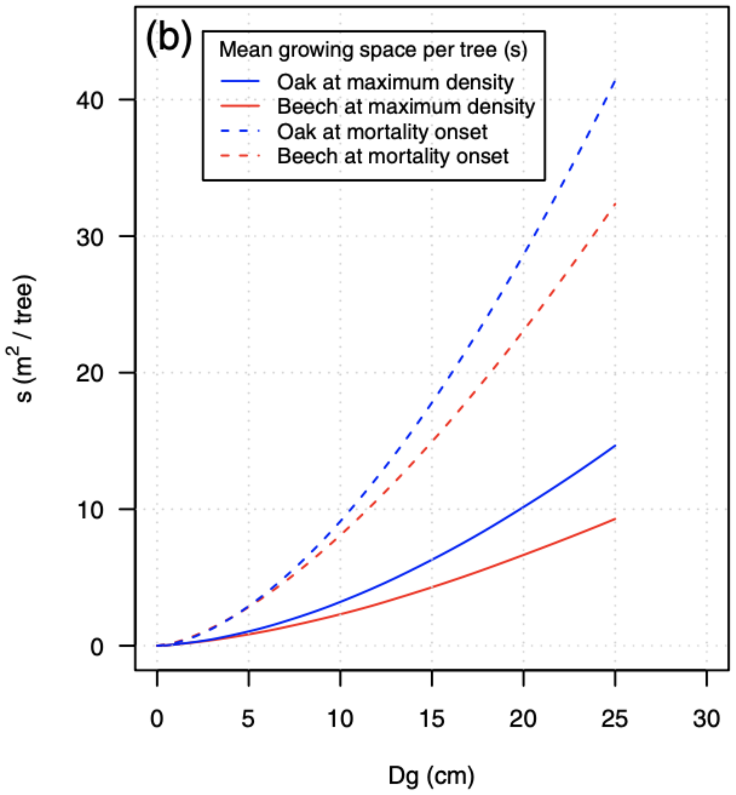
Comparison for a same initial density



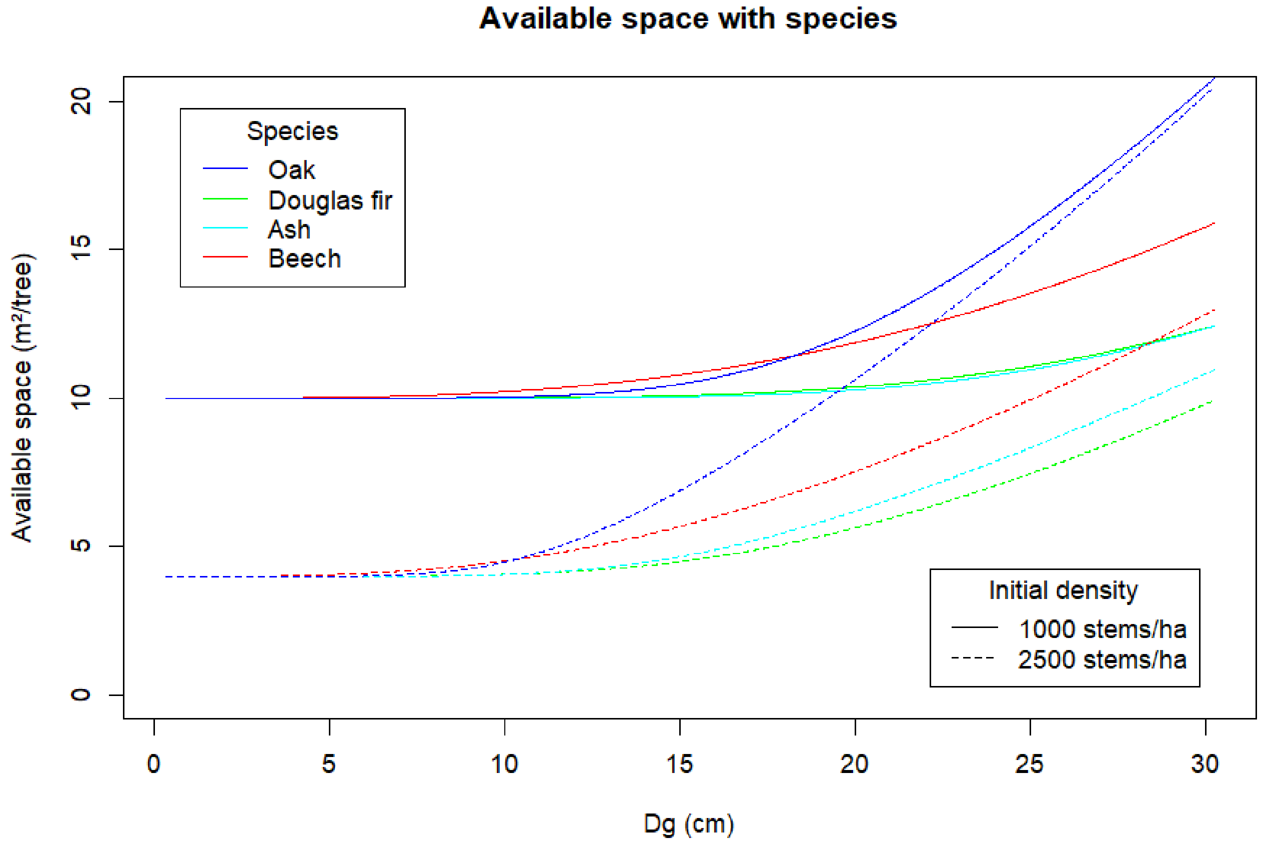
In terms of development stage, Oak is affected latter, but stronger, than Beech by intraspecific competition.

Application to growing space studies

Ningre et al. (2019) have used the same data to establish the curve of space needed by individuals of each species.



Curves obtained from fitted self-thinning equations



Although the point of view is not exactly the same, the results are comparable.

Conclusion

The modified Kurinobu and Miyaura approach allows to unify the models available in the literacy.

It depends on 4 parameters with an ecological interpretation :

- Initial density of the stand N_0 ;
- Slope a of the the Maximum Density Line: characteristic of assimilation apparatus;
- Intercept b of the Maximum Density Line: plot fertility ;
- Parameter of curvature c : sensitivity to competition.

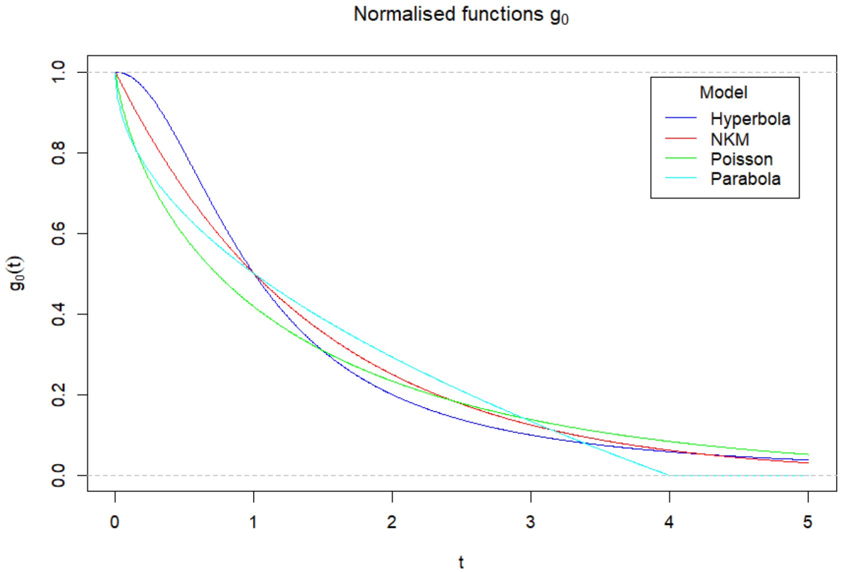
The models differ by the choice of the g function. As soon as these functions respect general properties, the corresponding models give very comparable results.

They mainly differ by the speed at which the trajectory tends to its asymptotes

The g function can be interpreted as the evolution of competition pressure with stand density (measured with the RDI).

The obtained trajectories are a sequence of equilibriums between the stand density and its development stage. The growing speed is not modelled.

Perspectives



Is there an ecological reasoning that would lead to a particular *g* function?

Can the modified Kurinobu and Miyaura approach be adapted to plurispecific and/or uneven-aged stands?

Thank you for attention